Analytical Theory of Transonic Normal Shock-Turbulent Boundary-Layer Interaction

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This paper describes an approximate analytical theory of the nonseparating transonic interaction of a weak normal shock with an isobaric laminar or turbulent boundary layer. The complicated multisubregion disturbance field structure is simplified by appropriate approximations (including neglect of the small scale details within the shock structure regions) which reduce the problem to an analytical form, provided the incoming Mach number is not very near unity. The resulting mixed transonic flow boundary value problem is solved by Fourier transformation methods. Lateral pressure gradient effects are included. Results are presented for turbulent boundary-layer pressure distributions, flow geometry, and skin friction. Following a small post-shock subsonic expansion region, the approach of the downstream interaction pressure to the final subsonic value is found to be a slow algebraic (1/*x*)-type decay in agreement with experimental observations.

Nomenclature

a	= variable introduced in Eq. (10a)
b	= constant defined in Eq. (9)
c	= constant defined in Eq. (9)
C_f	= local skin friction
Ci	= cosine integral
d_n	= numerical coefficient, see Eq. (11b)
D	= denominator function defined in Eq. (9)
Ei	= exponential integral
f_n, g_n	= numerical coefficients in Eqs. (12)
F, G	= combination of special functions defined in Eqs. (12)
h_i	$= p_i'/p_{0i} (i = 1, 2, 3)$
H_i	= Fourier transform of h_i
k	= wave number in Fourier transform
ℓ	= sublayer thickness ratio defined by Eq. (7)
p	= static pressure
Q	= unit function defined in Eq. (6a)
Re	= Reynolds number based on L
Si	= sine integral
u, v	= x- and y-velocity components
x, y	= corrdinates parallel and perpendicular to freestream
β	$= (M^2 - 1)^{1/2}$ or $(1 - M^2)^{1/2}$
γ	= ratio of specific heats
δ_{ij}	= Kronecker delta
δ_{BL}	= boundary-layer thickness
ΔP	= basic static pressure jump across shock
η	= displaced interface location
μ	= viscosity coefficient
π	= Fourier transform of pressure disturbance function
ρ	= density
ω	= exponent for power law viscosity law ($\mu \sim T^{\omega}$)

Superscripts

( )′	= disturbance quantity

Subscripts

e	= freestream properties upstream of shock at boundary-layer edge
0	= denotes undisturbed (not stagnation) flow property
L	= based on running length
w	= property at wall value
i	$= i = 1, 2, 3$, denotes region of variable
LS	= based on Lighthill sublayer

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I. Introduction

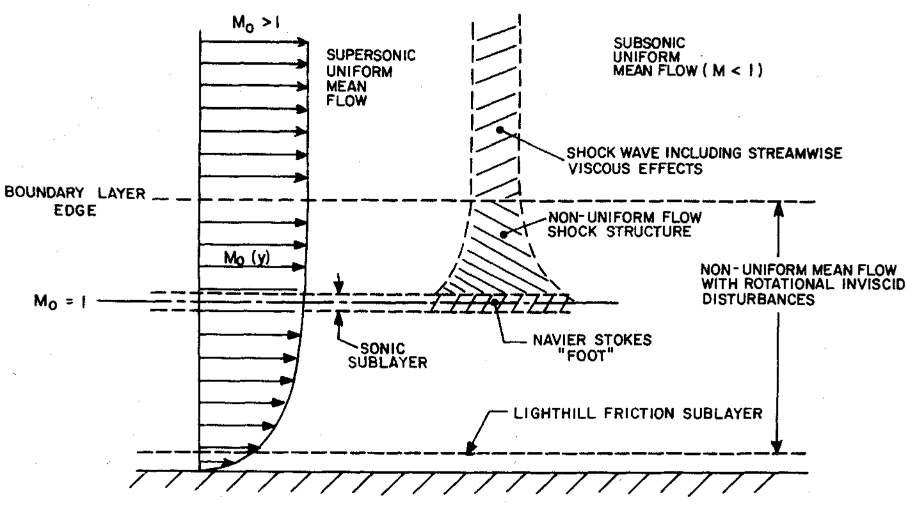
THE study of the interaction between an impinging shock wave and a boundary layer finds important application in transonic aerodynamics (e.g., understanding Reynolds number scaling effects and evaluating the off-design performance of "supercritical" wing sections), in the fluid mechanical problems of turbomachinery, and in the evaluation of boundary-layer effects within propulsion device inlets and diffusors. While many analytical investigations have been made of the laminar weak oblique shock case with supersonic post-shock flow,[1-3] relatively little has been done for the more difficult mixed-flow case of a strong normal shock wave with subsonic post-shock flow with a turbulent boundary layer. However, it is precisely this normal shock interaction problem which is of greatest practical interest in the aforementioned applications; it is therefore particularly desirable to devise an analytical approach for it, even if approximate, to guide the interpretation of experimental data and the subsequent development of more sophisticated theoretical models.

When a normal shock impinges on a laminar boundary layer, it is well-known that unless the shock is extremely weak (i.e., $M_l \leq 1.05$) the imposed adverse pressure gradient extensively separates the boundary layer,[4-6] causing a bifurcated shock wave pattern and a complicated highly- rotational region of mixed sub- and supersonic-flow behind the shock. Such a flow configuration is at present prohibitively difficult to treat theoretically. On the other hand, a turbulent boundary layer resists separation up to considerably higher shock strengths, covering a more useful practical Mach number range ($M_l \leq 1.3$); moreover, the resulting unseparated interaction involves no shock bifurcation, but instead exhibits a much simpler shock-decay weak interaction pattern[4-7] that is much more amenable to detailed theoretical analysis. Only a few analytic investigations have been conducted for the case of a supersonic flow terminated by a normal shock wave. Gadd[7] performed an experimental study of flow in a tube and a rough integral-method analysis of the boundary layer, while Melnik and Grossman[8] investigated normal shock-boundary layer interaction in more detail, using a combination of asymptotic and numerical methods that is quite different from the present approach.

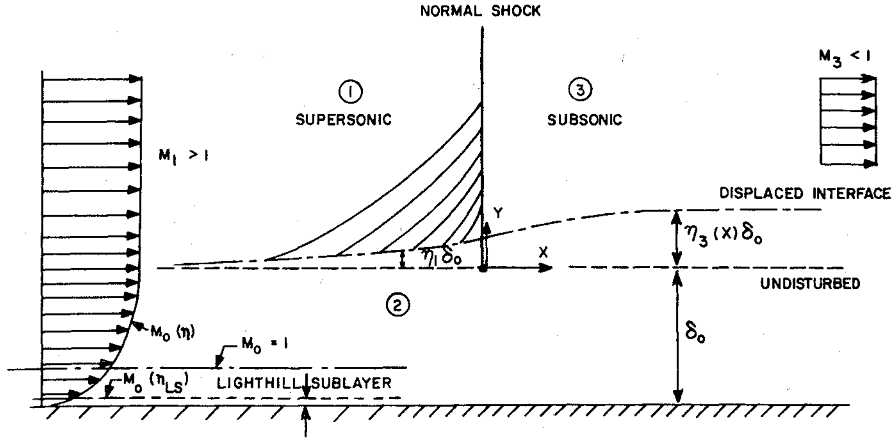
In the present paper, which is an abbreviation of Ref. 9, we consider the nonseparating transonic interaction of a weak normal shock with an isobaric turbulent boundary-layer flow, using an analytical approach based on an appropriate idealized, flow model which is the transonic normal shock-mixed flow counterpart (so to speak) of Lighthill's work. In Sec. II, the complicated multisubregion flow structure is simplified by the introduction of appropriate approximations that reduce the problem to an analytically tractible form provided M_1 is not very close to unity. The resulting approximate flow model embodies the essential qualitative physical features of *mixed* transonic flow found in nonseparating normal shock-turbulent boundary-layer interactions, including the lateral pressure gradient effect. In Sec. III, the solution of the attendant mathematical boundary value problem for the disturbance field is carried out by means of Fourier transformation methods. Section IV presents some typical numerical results for turbulent boundary-layer pressure distributions, disturbance flow geometry, and skin friction including some qualitative comparisons with available experimental data.

II. Formulation of Analytical Model

We consider the two-dimensional, compressible, high Reynolds number viscous flow of a thermally and caloricaly-perfect gas, with the basic equations expressed in appropriate nondimensional variables (see Nomenclature). In particular, the flow is taken to consist of an undisturbed parallel isobaric compressible shear flow $u_0(y)$, $\rho_0(y)$, $p_0=$constant [which simulates an unseparated laminar or turbulent boundary layer profile] subjected to small transonic disturbances u',v',ρ',p' due to the impingement of a weak normal shock. Then, in the leading approximation for nonseparating high Reynolds number flows, a stretched-variable asymptotic analysis[9] of the viscous compressible disturbance flow equations establishes the multiregional disturbance field structure schematically illustrated in Fig. 1a. The bulk of the flow consists of inviscid transonic perturbations external to or within the nonuniform boundary-layer profile. Imbedded within is a thin shock structure region consisting of a locally 1-D normal shock near the boundary-layer edge, a more general 2-D shock deeper down, and a thin horizontal sonic region layer representing the local upstream-spreading out of the shock as $M_0(y)\to 1$. Near the base of the shock is also a tiny Navier-Stokes transition region. Finally, there is the well-known thin viscous disturbance region or Lighthill sublayer near the wall wherein the no-slip condition is important.



A. ASYMPTOTIC FLOW STRUCTURE



B. APPROXIMATE FLOW MODEL

Fig. 1 Interaction flow model (schematic).

It thus appears that even in the leading approximation a rigorous detailed analysis of the present interaction problem is a complicated and formidable task involving the solution and matching of numerous horizontal and vertical subregions. If a reasonably tractible engineering analysis is desired, further simplying approximations that retain only the dominant physical features must be now introduced. Guided by the foregoing asymptotic analysis results plus heuristic physical considerations and the study of a very idealized version of the present problem involving a simple stepped boundary-layer profile,[9] the following simplifications of the flow model will thus be made for purposes of a first approximation.

a) Since most of the disturbance field is the inviscid flow ahead of, behind, and below the shock, we neglect the details within the shock structure regions by treating the incident shock and its extension down to the sonic line as a simple discontinuity across which Hugoniot shock jump relations are satisfied, while neglecting altogether both the Navier-Stokes region at the foot and the thin upstream sublayer along the sonic line. Aside from their very small size relative to the boundary-layer thickness, it can be shown[9] that these shock structure regions serve only to locally smooth the fine details of the flow and do not significantly influence the surrounding flow in the leading approximation unless M_1 is very close to unity.

b) The nonlinear transonic terms in the outer inviscid flow regions will be neglected, since in the present model much of their effect is already accounted for in the shock jump relations across the shock discontinuity. Moreover, Lighthill's work[2] contained a sonic point in the nonuniform mean flow, as did the transonic nonuniform mean flow cascade analysis given by Namba,[10] and these linearized solutions both passed through this point without any problem. It thus appears that when the explicit nonuniform mean flow term is retained in the governing equations, it is possible to neglect the nonlinear term without incurring difficulties, provided the external incoming Mach number is not very close to unity. This restriction is not too severe in turbulent flows because the present model is applicable up to Mach numbers of approximately 1.3 where incipient separation occurs.

c) We further simplify the problem by imposing the incident normal shock jump conditions only at the boundary-layer edge and neglecting the details of the subsequent shock penetration into the underlying nonuniform flow region. This idealization follows Lighthill's treatment of oblique shock interaction in which the weak shock wave is assumed to impinge on the boundary layer but the explicit effects resulting from the shock entering the nonuniform flow region below are neglected.[2] Although it introduces some minor inaccuracies in the pressure solution,[9] the resulting analytical simplications and good overall account of the interaction obtained fully justifies its use at this stage of development. Indeed, since the shock pressure jump at the boundary-layer edge is explicitly prescribed while below the sonic height the governing equation is of eliptic type so that no discontinuities can exist, the natural decay of the shock across the supersonic nonuniform flow region still tends to be largely simulated by this approximation.

d) In treating the Lighthill viscous sublayer (as we must to properly account for upstream influence effects), it is assumed thin enough to lie within the linear portion of the basic flow velocity profile. Although this introduces a modest error for turbulent flows, it has little effect on the global interaction predictions and is satisfactory for purposes of a first approximation.

The simplified flow model resulting from these approximations is summarized schematically in Fig. 1b. It involves an upper mixed flow region outside the boundary layer consisting of an incoming potential supersonic flow "1" and a subsonic potential flow "3" separated by a given weak normal shock, underlaid by a doubly-infinite nonuniform boundary-layer region "2" that contains a highly rotational, mixed, transonic linear disturbance flow. Near the wall there is the Lighthill viscous sublayer. These outer and inner regions are imagined to be separated along the boundary-layer edge by an interface whose perturbed location η is an unknown to be determined using the conditions that both pressure and streamline slope must be continuous across it. Provided the incident shock is weak enough to avoid a lambda-shock interaction pattern (which invalidates the present approach) while M_1 is not so close to unity that the incident shock thickness becomes a significant fraction of the boundary-layer thickness, this model contains the essential global features of the *mixed* transonic character of the nonseparating normal shock-boundary layer interaction problem. Moreover, at the expense of a possible localized singularity at the foot of the shock, the linearized theory involved is amenable to straightforward analytical treatment throughout the boundary layer; in particular we may adopt powerful Fourier transform methods.

III. Method of Solution

A. Regional Solutions

The linearized small disturbances in supersonic region "1" (Fig. 1b) are governed by the wave equation. Thus, if $\eta = \eta_1(x)$ is the small vertical displacement of the interface due to shock interaction, and incoming wave disturbances are ruled out, we have

$$p_1'(x,y) \simeq (\rho_{01} U_{01}^2/\beta_1)\frac{\partial \eta_1}{\partial x} \tag{1}$$

where $\beta_1 \equiv (M_1^2 - 1)^{1/2}$ and $\eta_1 = \eta_1(x - \beta_1 y)$.

The subsonic disturbance flow in quadrant "3" is caused by the interaction-generated interface displacement $\eta_3(x)$ along $y = \delta$, plus the post-shock perturbations along $x = 0^+$ resulting from the impingement of region 1 Mach wave disturbances on the shock, as given by Adams' shock-jump relation[11]

$$\frac{p_3(0^+,y)}{p_{03}} = -\left(\frac{M_3}{M_1}\right)\frac{p_1'(0^-,y)}{p_{01}} \tag{2}$$

where in the present transonic problem $M_1 \approx 1 + 0(\epsilon)$, $M_3 \approx 1 - 0(\epsilon)$ with $\epsilon << 1$ a given small number. Note from Eq. (2) that small supersonic compression disturbances ahead of the shock generate a post-shock subsonic expansion of about the same magnitude. Then, the solution in region 3 can be found[9] by Fourier Sine transformation as

$$\frac{p_3'(x,y)}{p_{03}} = \frac{2}{\pi}\int_0^\infty P_3(\hat{y},k)\sin kx\,dk \tag{3a}$$

where $\hat{y} \equiv y - \delta$ and

$$p_3(\hat{y},k) = e^{-k\beta_3\hat{y}}\int_0^\infty \left[\frac{\partial}{\partial y}\left(\frac{p_3'}{p_{03}}\right)\right]_{x,0^+}\left(\frac{\sin k\bar{x}}{k\beta_3}\right)dx$$
$$+\frac{M_3^2}{M_1^2}\beta_3\left[\cosh k\beta_3\hat{y}\int_0^\infty \frac{M_3^2}{M_1^2}\frac{p_1'}{p_{01}}(-\beta_1\eta)e^{-k\beta_3\eta}d\eta\right.$$
$$\left.+\int_0^y \frac{p_1'}{p_{01}}(-\beta_1\eta)\sinh k\beta_3(\eta-\hat{y})\,d\eta\right] \tag{3b}$$

with $\beta_3 = (1 - M_3^2)^{1/2}$. The first and second terms on the RHS correspond to the interface deflection and normal shock-crossing effects, respectively.

Within the rotational inviscid disturbance boundary-layer region "2", the Fourier transform $\pi(k,y)$ of the nondimensional disturbance pressure $p_2'/p_{02}(x,y)$ is governed by the Lighthill equation[2]

$$\frac{d^2\pi}{d\eta^2} - \frac{2}{M_0}\left(\frac{dM_0}{d\eta}\right)\frac{d\pi}{d\eta} - k^2(1 - M_0^2)\pi = 0 \tag{4}$$

where $M_0(\eta)$ is the boundary-layer Mach number profile and $\eta \equiv y/\delta$. The solution of Eq. (4) is subject to the zero normal flow inner boundary condition $(d\pi/d\eta)\eta_{SL}=0$ along the effective wall $\eta=\eta_{SL}$ defined by the displacement effect of the Lighthill friction sublayer, and the following outer boundary condition obtained by requiring that the upper and lower region pressure solutions match all along the interface:

$$\pi(k,\eta=1)=H_2(k)+\frac{\Delta p/p_{01}}{ik}+\frac{p_{03}}{p_{01}}H_3(k) \tag{5}$$

where $\Delta p = p_{03}-p_{01}$ is the given basic normal shock pressure jump (far downstream we require $p_3' \to 0$) while $H_2(k)$ is the Fourier transform of $h_2(x) \equiv p_2'(x,1)/p_{02}$ and we define the "half-range" transforms

$$H_1(k) \equiv \int_{-\infty}^{0} h_1 e^{-ikx}dx$$

and

$$H_3(k) \equiv \int_{0}^{\infty} h_3 e^{-ikx}dx$$

Then, introducing the "unit" solution $Q(k,\eta)$ of Eq. (4) which satisfies the wall boundary conditions $Q(k,\eta_{SL})=1$, $dQ/d\eta\,(k,\eta_{SL})=0$ [which is found by straight-forward outward numerical integration for any given $M_0(y)$ profile], the general solution subject to the stated boundary conditions is simply

$$\pi(k,\eta)=\frac{Q(k,\eta)}{Q(k,1)}H_2(k) \tag{6a}$$

or in physical variables

$$\frac{p_2'(x,\eta)}{p_{02}}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{Q(k,\eta)H_2(k)}{Q(k,1)}e^{ikx}dk \tag{6b}$$

Within the viscous sublayer, the solution for a linear basic flow profile $U_0 \simeq (\tau_w/\mu_w)y$ is already well known[2] and need be only briefly outlined here. Upon Fourier transformation with respect to x, the normal velocity disturbance equation in the layer is found to be[2]

$$\frac{d^4\tilde{V}(k,\eta)}{d\eta^4}=\frac{\eta}{\ell^3}\frac{d^2\tilde{V}}{d\eta^2} \tag{7}$$

where

$$\ell=(T_w/T_e)^{(1+\omega)/3}/[ikRe_\delta(dU_0/d\eta)_w]^{1/3}$$

is a characteristic ratio of sublayer to boundary-layer thickness. Equation (7) defines $d^2V/d\eta^2$ in terms of Airy functions, which upon two subsequent outward integrations, application of the zero mass transfer $[\tilde{V}(0)=0]$ and no slip $[d\tilde{V}/d\eta(0)=0]$ conditions at the surface, and discarding the exponentially-divergent solution yields the $\tilde{V}$-field and the following result for its effective "displacement thickness"

$$\eta_{LS}=0.776\left(\frac{1}{\kappa}\right)^{1/3}\left[\frac{1}{R_{e_\delta}{}^2}\frac{2}{Cf}\left(\frac{T_w}{T_e}\right)^{1+2\omega}\right]^{1/3} \tag{8}$$

where κ is the smallest zero of the function $D(1,\kappa)$ defined below, corresponding to the large scale upstream influence.

B. Global Interaction Relations

Applying the foregoing regional solutions to the interfacial matching conditions of continuous pressure and streamline slope v_1/U_0 at the boundary-layer edge now enables the derivation of basic integro-differential equations for the interaction pressure along the interface. *Upstream of the shock*, it is found that[9]

$$H_1(K)=\frac{-\left(\frac{dQ}{d\eta}\right)_{K,1}\left(\frac{\Delta p}{p_{01}}\right)}{iKD(K,1)}-\frac{b}{\pi}\frac{(K/|K|)Q_\eta(K,1)}{D(K,1)}$$
$$\times\int_{-\infty}^{+\infty}\left[\frac{\xi}{C(K^2-\xi^2)}+\frac{ic}{C|K|+i\xi}\right]H_1(\xi)d\xi \tag{9}$$

where

$$b=p_{03}M_3{}^2/p_{01}M_1{}^2, \lambda=\beta_3/\beta_1$$

while the denominator function $D(K,1) \equiv [(dQ/d\eta)+ik\beta_1 Q]_{\eta=1}$ is the same quantity that appeared in Lighthill's analysis[2] of the purely supersonic shock interaction problem. The first term on the RHS of Eq. (9) is due to the basic pressure jump across the normal shock wave (the forcing function of the problem), while the second integral term is due to the elliptic mathematical nature of the subsonic flow behind the shock.

Since the inversion of $H_1(k)$ back to $h_1(x)$ pertains to $x<0$, we desire the solution of Eq. (9) on the lower complex plane $I_m(k)<0$. Then, introducing the new dependent variable

$$a(k) \equiv \frac{ikH_1(k)}{(\Delta p/p_{01})(dQ/d\eta)(k,1)} \tag{10a}$$

which contains no singularities on the negative imaginary axis, Eq. (9) reads

$$a(k)=-1-\frac{b}{\pi}\frac{k^2}{|k|}\int_{-\infty}^{+\infty}\left\{\frac{\xi}{c(k^2-\xi^2)}+\frac{ic}{c|k|+i\xi}\right\}$$
$$\frac{Q_\eta(\xi,1)a(\xi)}{\xi D(k,1)}d\xi \tag{10b}$$

Once Eq. (10b) is solved (see the following), the physical plane solution can then be obtained by the Fourier inversion integral using the method of residues, with the following result[9]:

$$\frac{h_1(x)}{\Delta p/p_{01}}=\sum_{n=1}^{\infty}d_n e^{\kappa_n x},\ x<0 \tag{11a}$$

where

$$d_n=\frac{Q_\eta(-i\kappa_{n,1})a(-i\kappa_n)}{\kappa_n\frac{d}{d\kappa}[Q_\eta(-i\kappa_n,1)+\beta_1 Q(-i\kappa_n,1)]} \tag{11b}$$

and where κ_n are the zeros of $D(1,k)$ as found by direct analysis of the unit solution Q to Eq. (4). Equation (11a) describes an exponential pressure decay with upstream distance with the characteristic upstream distance scale $\sim(\kappa_1)^{-1}$ as discussed by Lighthill.[2]

Downstream of the shock, a similar but more lengthy analysis[9] yields the following interface pressure solution:

$$\frac{h_3(x)}{\Delta p/p_{01}}=\frac{-2/\pi}{(M_1/M_3)^2}\left[\frac{1}{2\beta_3}\sum_{n=1}^{\infty}\frac{f_n}{g_n}F(x)+\sum_{n=1}^{\infty}d_n G\left(\frac{\kappa_n X}{c}\right)\right] \tag{12}$$

where

$$f_n=Q_\eta(ig_n,\eta=1)/g_n Q(ig_n,\eta=1),$$

$$F(x)=e^{-x}Ei(x)-e^{x}Ei(-x)$$

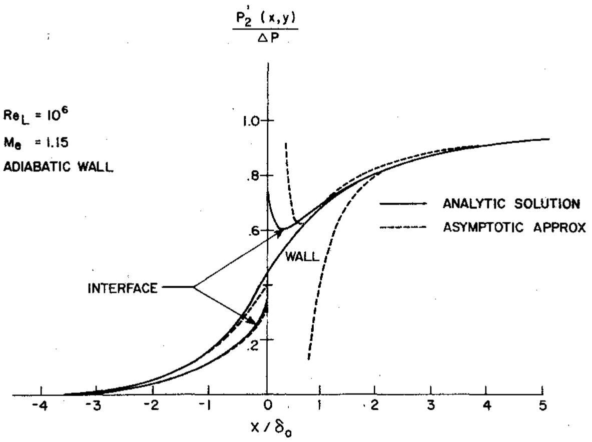


Fig. 2 Typical interaction pressure distributions.

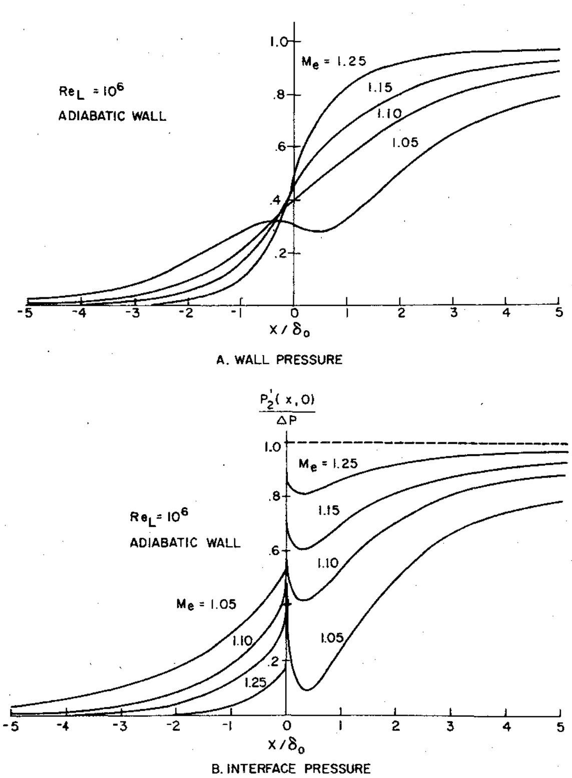


Fig. 3 Effect of shock strength on interaction.

and

$$G(x) = Ci(x)\sin x - Si(x)\cos x$$

Equation (12) predicts an algebraic decay in the pressure perturbation p_3' far downstream of the type $p_3' \sim x^{-1}$.

C. Surface Properties

With the disturbance pressure distribution known along the interface, the remaining disturbance field properties can be readily found. Of particular practical and experimental interest are the forces acting along the surface. Using the convolution theorm, Eq. (6b) can be used to obtain a lengthy analytical expression for the wall pressure[9]; in particular, far downstream of the shock ($x >> 1$) it predicts an algebraic asymptotic behavior that decays as x^{-1}

$$\frac{p_2'(x,0)}{\Delta p} \simeq 1 + \frac{2p_{03}/p_{01}}{(M_1/M_3)^2}\left[\frac{B_1}{x} + 0(x^{-3})\right] \tag{13}$$

where B_1 is a constant.

Under the present simplifying assumptions, Ingers solution[12] for the interaction-induced shear stress perturbation within the Lighthill viscous sublayer can be carried over directly to the present problem; in terms of the surface pressure solution it gives

$$C_f'(x) = \frac{-3.864\mu_{ow}\int_{-\infty}^{\infty}\pi(k,0)e^{ikx}dx}{\pi\rho_{01}\rho_{wo}U_{01}(dU_0/dy)_w\delta\ell^2} \tag{14}$$

Although the integral in Eq. (12) involves a branch point at the origin, which makes it very difficult to evaluate for the upper complex plane values required for a downstream solution ($x>0$), it can be readily evaluated by residues for the *lower* complex plane pertaining to $x<0$ upstream of the shock and the result expressed in terms of p_w'.

IV. Discussion of Results

A. Evaluation Procedure

The solution of the pressure perturbation, Eq. (4), for the unit solution $\pi = Q(k,\eta)$ was obtained by straight-forward outward numerical integration. The Mach number profile $M_0(y)$ was evaluated using the analytical turbulent profile of Inger and Williams[13] which is particularly well-adapted for this purpose. By running such calculations with the wave number $k = -i\kappa$ as a parameter, the various zeros of the functions $Q(k,1)$ and $D(k,1)$ can be determined.

Turning to the solution of the basic upstream interface interaction Eq. (9), the integrals therein can be expressed in terms of summations over the residues of their integrands assuming $a(k)$ has no singularities. Introducing the real variable ξ such that $\kappa = -i\xi$, one obtains the set of first order differential equations

$$\sum_{n=1}^{\infty}\left\{\delta_{mn}\left[1 + \frac{b}{CD_\xi}\left\{Q_{\eta\xi}(\xi_m) - \frac{Q_\eta(\xi_m)}{2\xi_m}\right\} - \frac{2cb}{(C+1)\xi_m}\frac{Q_\eta(\xi_m)}{D_\xi(\xi_m)}\right] - (1-\delta_{mn})\left[\frac{c^{-1}}{\xi_m{}^2 - \xi_n{}^2} + \frac{c}{\xi_n(c\xi_m + \xi_n)}\right]2b\xi_m\right\}a(\xi_m)$$
$$= -1 - \frac{b}{c}\frac{Q_\eta(\xi_m)}{D_\xi(\xi_m)}\frac{\mathrm{d}a}{\mathrm{d}\xi}(\xi_n) \tag{15}$$

where $n = 1, 2, \ldots, m$ and where the gradient term on the RHS arises because of a second-order pole at $n=m$ in one of the original integrands. A standard finite difference approach[14] was then used to solve Eq. 15 for the $a(k)$, truncation to three of four terms usually being sufficiently accurate.

B. Typical Results

The major features of the resulting interaction pressure field solution are illustrated in Fig. 2, where both the boundary-layer edge and wall pressure distributions are shown for a typical case. It is seen that the shock-induced lateral pressure gradients are significant within a region of several boundary-layer thicknesses upstream and downstream of the shock foot, the wall pressure being higher than the edge ahead of the shock and lower behind the shock. Upstream of the shock, these pressures decay exponentially with distance. Along the boundary-layer edge, a local pressure jump occurs across the shock at $x=0$ followed by a small region of subsonic post-shock expansion and subsequent recompression, in agreement with experimental observations.[4,7] One boundary-layer thickness or more downstream, the wall and edge pressures have equalized and rise monotonically to the final post-shock level, asymptoticaly approaching it algebraically like x^{-1}.

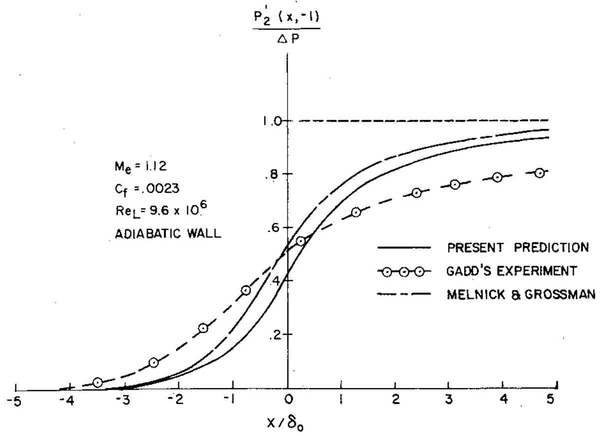


Fig. 4 Comparisons with experiment (unseparated flow).

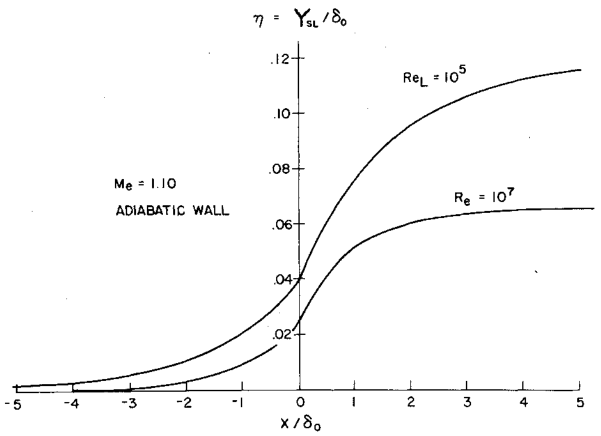


Fig. 5 Interaction-induced boundary-layer thickening.

The effect of incoming Mach number (incident shock strength) on the interaction is shown in Fig. 3. As M_1 increases, the strength of the outer local shock jump increases while the upstream interaction and upstream influence distance decline, as observed experimentally.[4,7] Near sonic conditions $M_1 \to 1$, comparison of Figs. 3a and 3b shows that the effect of the post-shock expansion tends to persist well inward across the boundary layer, again in qualitative agreement with observation.[4]

Figure 4 shows a comparison between the wall pressure predictions of the present approximate theory, the more elaborate numerical solutions obtained by Melnick and Grossman[8] and the pipe flow experimental data of Gadd[7] at $M = 1.12$ and $Re_L = 8.6 \times 10^6$. It is seen that these theories are in good agreement with each other and possess virtually the same accuracy compared to experiment. The theory moderately underestimates the upstream influence and overestimates the rate of post-shock pressure recovery; in particular, the predicted recovery of the full incident shock pressure jump far downstream is only partially obtained (80%) in Gadd's experimental data. This lack of full recovery is thought to be associated with the noninteractive downstream nonequilibrium relaxation of turbulent wall boundary layers, and is under further study.

Using the pressure solutions in the y-momentum equation and then integrating twice with respect to x along $y = \delta$ yields the following expressions[9] for the interface displacement (boundary-layer thickness change) $\eta(x)$:

$$\eta(x) = \frac{\Delta P}{p_{01}} \frac{\beta_e}{\gamma M_e^2} \sum_{n=1}^{\infty} \frac{d_n e^{\kappa_n x}}{\kappa_n}, \quad x<0 \tag{16a}$$

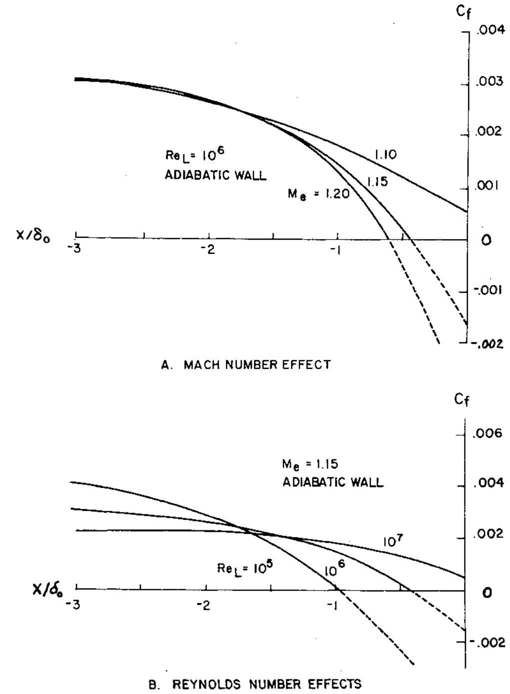


Fig. 6 Interaction effect on local skin friction.

$$= \eta(0) - \frac{\pi}{2} \frac{\Delta P/p_{01}}{\gamma M_1^2} \sum_{n=1}^{\infty} \frac{f_n}{g_n^2} (e^{-g_n x} - 1), \quad x>0 \tag{16b}$$

The typical shape of the interaction-deflected outer streamline is shown in Fig. 5, and illustrates the expected thinning out with increasing Reynolds number. Note that the streamline *slope* is discontinuous across the shock due to the simplifying approximations in the present theory.

Corresponding to this outer streamline deflection, there is also a small horizontal perturbation in the local shock location outside the boundary layer from its undisturbed position $x = 0$. Although at first sight determinable from an integration of the v_3' solution using an equation given by Adams,[11] this perturbation was in fact found to involve an indeterminate integral to first order at large y owing to the slow algebraic decay of the disturbance field along the semi-infinite downstream interface. Thus, second-order terms must be considered to determine the perturbed shock location in the present problem, this being a common occurance in the linearized supersonic flow theory as discussed by Hayes.[15]

As illustrated in Figs. 6a and 6b, the upstream skin friction decreases toward the shock owing to the adverse pressure gradient induced by the shock-boundary-layer interaction. Indeed, the present analysis predicts vanishing skin friction in some cases. Although not valid for separated flow ($Cf \leq 0$), these analytical results nevertheless are useful to infer some basic qualitative trends. Thus, we see that incipient separation moves upstream with either increasing M_1 or decreasing Re_L, both of these trends being in agreement with the experimental evidence.[6,7]

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